Polynomial and Rational Inequalities

Steps for Solving Polynomial and Rational Inequalities

1. Write the inequality so that a polynomial or rational expression f is on the left side and zero is on the right side in one of the following forms:

   $f(x)>0$ $\qquad$ $f(x)<0$ $\qquad$ $f(x)\geq 0$ $\qquad$ $f(x)\leq 0$

   For rational expressions, be sure that the left side is written as a *single quotient* and find its domain.
2. Determine the real numbers at which the expression f on the left side equals zero and, if the expression is rational, the real numbers at which the expression f on the left side is undefined. (In other words, find any *x*-intercepts and vertical asymptotes of the graph of $y=f(x)$.
3. Use the numbers found in Step 2 to separate the *x*-axis into intervals.
4. Determine whether the graph of $y=f(x)$ is above or below the *x*-axis in each interval.

   The solutions of $f(x)>0$ are the *x*-values for which the graph is **above** the *x*-axis.

   The solutions of $f(x)\geq 0$ are the *x*-values for which the graph is **on or above** the *x*-axis.

   The solutions of $f(x)<0$ are the *x*-values for which the graph is **below** the *x*-axis.

   The solutions of $f(x)\leq 0$ are the *x*-values for which the graph is **on or below** the *x*-axis.

   ★ **Warning:** For $f(x)\geq 0$ and $f(x)\leq 0$, include the *x*-intercepts *but not the asymptotes* in the solution set.

-OR-

4. Select a number in each interval and evaluate $f(x)$ at the number.
   a. If the value of $f(x)$ is positive, then $f(x)>0$ for all numbers x in the interval.
   b. If the value of $f(x)$ is negative, then $f(x)<0$ for all numbers x in the interval.

   The solution set is all the intervals that make the inequality true.

   ★ **Warning:** For $f(x)\geq 0$ and $f(x)\leq 0$, include the values for which $f(x)=0$ (the *x*-intercepts) *but not the values for which* $f(x)$ *is undefined* (the asymptotes) in the solution set.

Example: Solve the inequality $x^4>9x^2$, and graph the solution set.

$x^4-9x^2>0$

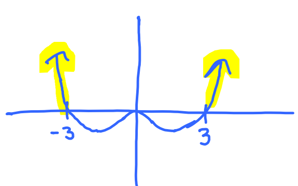
x-ints: $x^2(x^2-9)=0$

$x^2(x+3)(x-3)=0$

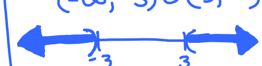
$x=0,-3,3$

touches crosses

end behavior: $y=x^4$ ↑ ↑



$(-\infty,-3)\cup(3,\infty)$



OR

Interval	$(-\infty,-3)$	$(-3,0)$	$(0,3)$	$(3,\infty)$
Number picked	-4	-1	1	4
f(number)	$(-4)^4-9(-4)^2$ $=112$	$(-1)^4-9(-1)$ $=-8$	$(1)^4-9(1)$ $=-8$	$(4)^4-9(4)^2$ $=112$
pos/neg	positive	negative	negative	positive

Example: Solve the inequality $x^4 \geq 8x$, and graph the solution set.

$x^4 - 8x \geq 0$

x-intercepts:

$x^4 - 8x = 0$

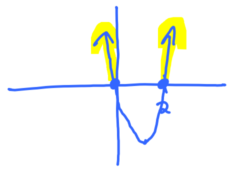
$x(x^3 - 8) = 0$

$x(x-2)(x^2+2x+2) = 0$

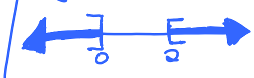
$x = 0, 2$ crosses

no real roots $b^2 - 4ac < 0$

end behavior: $y = x^4$ ↑ ↑



$(-\infty, 0] \cup [2, \infty)$



OR

Interval	$(-\infty, 0]$	$[0, 2]$	$[2, \infty)$
Number Picked	-1	1	3
f(number)	$(-1)^4 - 8(-1) = 9$	$1^4 - 8(1) = -7$	$3^4 - 8(3) = 57$
pos/neg	pos	neg	pos

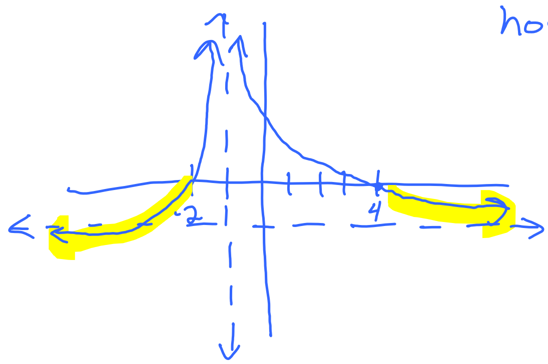
Example: Solve the inequality $\dfrac{(x+2)(4-x)}{(x+1)^2} < 0$, and graph the solution set.

x-ints: $(x+2)(4-x) = 0$

$x = -2 \quad x = 4$

vert. asymp: $(x+1)^2 = 0$

$x = -1$

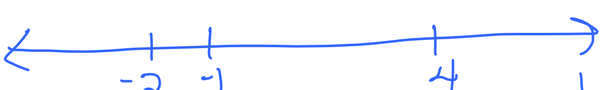


horiz. asymp:

$y = \dfrac{-x^2}{x^2}$

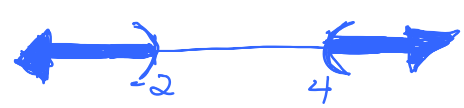
$y = -1$

- OR -



Interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 4)$	$(4, \infty)$
Number chosen	-3	-1.5	0	5
f(number)	$f(-3) = -\frac{7}{4}$	$f(-1.5) = 11$	$f(0) = 8$	$f(5) = -\frac{7}{36}$
pos/neg	neg	pos	pos	neg

$(-\infty, -2) \cup (4, \infty)$



Example: Solve the inequality $\frac{3x+5}{x+4} \le 2$, and graph the solution set.

$$\frac{3x+5}{x+4} - 2 \le 0$$

$$\frac{3x+5}{x+4} + \frac{-2(x+4)}{x+4} \le 0$$

$$\frac{3x+5-2x-8}{x+4} \le 0$$

$$\frac{x-3}{x+4} \le 0$$

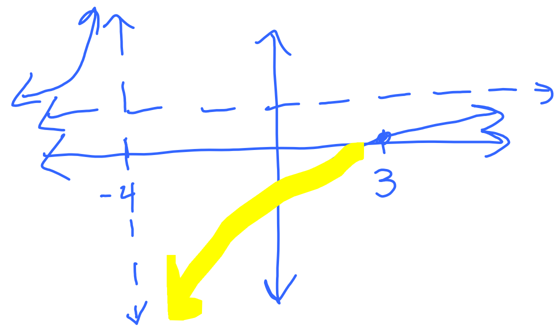
x-int: $x-3=0$

$x=3$

vert. asymp: $x+4=0$

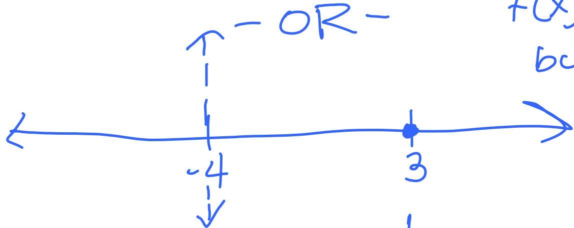
$x=-4$

horiz. asymp: $y=\frac{x}{x}$ $\quad y=1$



$f(x)=0$ at 3,
but $f(x)\neq 0$ (it's undefined)
at -4.

-OR-



Interval	$(-\infty,-4)$	$(-4,3]$	$[3,\infty)$
Number Chosen	-5	0	4
f(Number)	$\frac{-5-3}{-5+4} = 8$	$\frac{0-3}{0+4} = -3/4$	$\frac{4-3}{4+4} = \frac{1}{8}$
pos/neg	pos	neg	pos

$(-4, 3]$